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Analysis and Identification of Time-Delay Systems via Piecewise Linear Polynomial Functions

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Abstract. This paper extends the application of piecewise linear polynomial functions to the analysis and parameter identification of linear time-invariant continuous delayed systems. The real Schur from decomposition is used to structure the resulting algebraic equations in such a way that it can be solved columnwise by a forward substitution technique. The main advantage of this method is that a simple recursive algorithm is derived. Tedious iterative algorithms and direct matrix inversion for large scale systems are thus avoided in calculating the expansion coefficients. Illustrative examples are given for demonstration.

1. Introduction

Systems with time delay are met frequently in transmission lines, mechanical systems, industrial and chemical processes, neural networks and others. In recent years, analysis, state estimation and parameter identification of delayed systems have been established using orthogonal functions (OFs) (e.g. [1-7]).

The main characteristic of OFs is that both sides of a given set of state space equations are approximated by a finite dimensional subspace spanned by a chosen set or orthogonal vectors. By equating the coordinates of the finite subspace with respect to the chosen basis, a set of linear algebraic equations is obtained and can be solved by employing either techniques of matrix inversion or an iteration algorithm. Unfortunately, the usefulness of this technique is limited by two difficulties, the first is the analyticity requirements for both the input and output signals. The second difficulty is the appearance of severe oscillations at large intervals[8]. To alleviate the aforementioned drawback, Liou[9] suggested that the piecewise linear polynomial functions (PLPF) should be taken as a set of approximating basis functions and the interval should be divided into smaller pieces. A significant advantage of PLPF for the problems of analysis and parameter identification has been demonstrated in[10].

In this paper, the application of PLPF is extended to the analysis and parameter identification of linear time-invariant continuous delayed systems. Using the delay operational matrix of integration, a Sylvester-type matrix equation is obtained. A transformation method is used which employs the real Schur form decomposition[11] to structure the equation in such a way that it can be solved columnwise by a forward substitution technique. A recursive algorithm is derived and system equations can be solved with low dimensional matrix. Computational examples are included to illustrate the effectiveness of the proposed method.

2. Properties of PLPF

A piecewise linear polynomial function is defined as[9]:

$$
\left.
\begin{array}{l}
P_o(t) = \begin{cases} 1-(q/T)\,t, & 0 \le t \le (T/q) \\ 0, & \text{otherwise} \end{cases} \\
P_i(t) = \begin{cases} (1-i)+(q/T)\;t, & (i-1)\;(T/q) \le t \le i(T/q) \\ (1+i)-(q/T)\;t, & i(T/q) \le t \le (i+1)\;(T/q) \\ 0, & \text{otherwise} \end{cases} \\
\text{for}\;\; i = 1, 2, \;\ldots,\; m-2 \\
P_{m-1}(t) = \begin{cases} 0, & \text{otherwise} \\ (1-q)+(q/t)\;t, & (q-1)\;(T/q) \le t \le T \;, \end{cases} \\
\text{where}\;\; q = m-1.
\end{array}
\right| \tag{1}
$$

The basis functions {P_o, P_1, ... , P_{m-1} are linearly independent in the interval [0, T]. At the break point we have,

$$
P_i(jT/q) = \begin{cases} 1, & i = j \\ 0, & j \ne j \end{cases} \quad \text{for all}\;\; i, j \tag{2}
$$

Furthermore, it is very easy to show that

$$
\sum_{i=0}^{m-1} P_i(t) = L\;\; P(t) = 1 \;\;, \tag{3}
$$

where,

$$
L \;=\; [1\;1\;1\;................\;1], \text{ and} \tag{4}
$$

$$
P(t) \;=\; [P_o(t)\; P_1(t)\; ...\; P_{m-1}(t)^T. \tag{5}
$$

The arbitrary function f(t) can be approximated as:

$$f(t) = \sum_{i=0}^{m-1} f_i \; P_i(t) = F \; P(t) \;, \tag{6}$$

where,

$$F \;=\; [f_o \;\; f_1 \ldots f_{m-1}] \;. \tag{7}$$

The coefficients of the expansion are obtained from:

$$f_i \;=\; f(iT/q) \;. \tag{8}$$

The operational matrix of integration for the PLPF can be performed by the procedure given in[9], as follows:

$$\int_0^t P(s)\, ds = G\, P\,(t) \;. \tag{9}$$

Where G is called the operational matrix of integration and is defined as:

$$G = (T/q)\begin{bmatrix} 0 & 0.5 & 0.5 & 0.5 & \ldots & 0.5 \\ 0 & 0.5 & 1 & 1 & \ldots & 1 \\ 0 & 0 & 0.5 & 1 & \ldots & 1 \\ . & . & . & . & . & \\ . & . & . & . & . & \\ . & . & . & . & . & \\ 0 & 0 & 0 & 0 & \ldots & 0.5 \end{bmatrix} \tag{10}$$

The delay PLPF $P_i(t - jT/q)$ is defined as the shift of PLPF $P_i(t)$ along the time axis by J subintervals. It is straight forward to verify that the integration of $P(t - JT/q)$ can be performed by the procedure given in[9] as follows:

$$\int_0^t P(s - JT/q) \;\; ds \;\; = HP(t) \;, \tag{11}$$

where,

$$H \approx I[m; -J]G \;. \tag{12a}$$

$$I[m; -J] = \begin{bmatrix} 0 & I_{(m-J)} \\ 0 & 0 \end{bmatrix} \tag{12b}$$

$I_{(m-J)}$ is an $(m - J) \times (m - J)$ identity matrix.

In the next section, the concepts introduced above are applied to the analysis of linear time-invariant delayed systems

3. Analysis of Time-Delay Systems

Consider a linear time-invariant system with time delay described by the following state equation,

$$x(t) = A\, x(t-\alpha) + B\, u(t)\,. \tag{13}$$

Where $x(t) \in R^n$, $u(t) \in R^r$, and A, B are n x n and n x r constant matrices, respectively. α is a positive fixed delay. The initial conditions are:

$$x(0) = K\,, \tag{14}$$

$$x(t) = \phi(t) \quad \text{for } t\,,\, 0\ . \tag{15}$$

Where K is a constant vector and $\phi(t)$ is an arbitrary known time function.

Approximation of x(t), $\phi(t)$, and u(t) by PLPF of size m gives:

$$x(t) = \sum_{i=0}^{m-1} x_i\, P_i(t) = X\, P(t)\ , \tag{16}$$

$$\phi(t) = \sum_{i=0}^{m-1} \phi_i\, P_i(t) = \phi\, P(t)\ , \tag{17}$$

$$u(t) = \sum_{i=0}^{m-1} u_i\, P_i(t) = U\, P(t)\ , \tag{18}$$

where,

$$X = [x_o \ \ x_1 \ \ \dots \ \ x_{m-1}]\ , \tag{19}$$

$$\phi = [\phi_o \ \ \phi_1 \ \ \dots \ \ \phi_{m-1}]\ , \tag{20}$$

$$U = [u_o \ \ u_i \ \ \dots \ \ u_{m-1}]\ . \tag{21}$$

Here X, ϕ, and U are n x m, n x m, and r x m constant matrices, respectively. ϕ and U can be computed using equation (8), and X is to be determined.

In order to solve the state equation with time delay, equation (13) is integrated from 0 to t:

$$x(t) - x(0) = A\int_0^t x(s-\alpha)\ ds + B\int_0^t u(s)\ ds\ . \tag{22}$$

For the interval $0 \le t \le a$, equation (15) is used:

$$x(t-\alpha) = \phi(t-\alpha)\ . \tag{23}$$

Substituting equation (23) in equation (22) results in:

$$x(t) - x(0) = A\int_0^\alpha \phi(s-\alpha)\ ds + A\int_0^t x(s-\alpha)\ ds + B\int_0^t u(s)\ ds \tag{24}$$

Substituting equations (16) – (18) into equations (24) and applying (3), (9), (11), and (14) give:

$$XP(t) - KLP(t) = A\phi HP(\alpha)LP(t) + AXH[I\text{-}P(\alpha)L]P(t) + BUGP(t) \ , \qquad (25)$$

or, the algebraic matrix equation ,

$$X = A\,X\,W + S\ , \qquad (26)$$

where

$$W \ = \ H[I - P(a)\ L]\ , \qquad (27)$$

$$S \ = \ KL + A\ \phi\ HP\ (\alpha)\ L + BUG\ \ . \qquad (28)$$

The matrix $P(\alpha)$ is equivalent to the matrix $P(t)$ defined in equation (5) with $t = \alpha$.

Therefore, the solution of equation (26) is obtained by

$$\begin{bmatrix} x_0 \\ x_1 \\ . \\ . \\ . \\ . \\ x_{m-1} \end{bmatrix} = [I - (W^T \otimes A)]^{-1} \begin{bmatrix} s_0 \\ s_1 \\ . \\ . \\ . \\ . \\ s_{m-1} \end{bmatrix} \qquad (29)$$

where I is an mn x mn identity matrix and the operator $\otimes$ denotes the Kronecker product[12]. However, much more computer time is consumed in solving mn simultaneous equations, making this approach impractical except for small systems. Therefore, it is important to search for a simple recursive formula for solving equation (26). Fortunately, a recursive formula can be found by transforming W into its real Schur form decomposition[13]. The transformed system of equation (26) is given by

$$(XV) \ = \ A\ (XV)\ (V^T\ W\ V) + S\ V \qquad (30)$$

Where V is orthogonal matrix and $(V^T\ WV)$ is upper quasi-triangular matrix (A quasi-triangular matrix is triangular with possible nonzero 2×2 blocks along the diagonal). If we define $XV = Y$, $V^T\ W\ V = R$ and $SV = M$, the transformed system becomes :

$$Y = A\ Y\ R + M \qquad (31)$$

Assuming $r_{k+1,\ k}$ is zero, then if the K^{th} column of each side of (31) is taken, the following equation is yield:

$$y_k = A\left(\sum_{j=1}^{k} r_{j,k}\ \ y_j\right) + m_k\ , \quad k \ = \ 1,\ m\ , \qquad (32)$$

where

$$Y = [y_1\ \ y_2\ \ \ldots\ \ y_m]\ , \qquad (33)$$

$$M = [m_1 \ m_2 \ ... \ m_m] \quad , \tag{34}$$

Hence, y_k can be found by solving

$$(I - A\, r_{k,k})y_k = m_k + A\left(\sum_{j=1}^{k-1} r_{j,k}\, y_j\right) \tag{35}$$

Equation (35) can be solved columnwise by starting at column 1 and working forwards to column m. If $(I - A\, r_{k,k})$ is nonsingular, these linear systems of equations can be solved using Gaussian elimination with partial pivoting. As shown in equation (35), only n simultaneous equations need to be solved. Tedious direct method for solving mn simultaneous equations are avoided. Thus, the computer time is greatly reduced. However, if $r_{k+1,k} \neq 0$ (due to the presence of 2×2 bumps on the diagonal of R), then we have,

$$(I - A\, r_{k,k})\, y_k - r_{k+1,k}\, A y_{k+1} = m_k + A\left(\sum_{j=1}^{k-1} r_{j,k}\, y_j\right) , \tag{36}$$

Letting k = k + 1 in equation (36) and utilizing the fact that the quasi upper triangular matrix cannot have two consecutive nonzero elements along its subdiagonal, implies that $r_{k+2,\, k+1} = 0$, and equation (36) becomes

$$(I - A\, r_{k+1,k+1})\, y_{k+1} - r_{k,k+1}\, A y_k = m_{k+1} + A\left(\sum_{j=1}^{k-1} r_{j,\, k+1}\, y_j\right) , \tag{37}$$

Then equation (37) can be written as;

$$y_{k+1} = E_k\left(r_{k,k+1}\, A y_k + m_{k+1} + A\left(\sum_{j=1}^{k-1} r_{j,k+1}\, y_j\right)\right) , \tag{38}$$

where,

$$E_k = (I - A\, r_{k+1,\, K+1})^{-1} \quad , \tag{39}$$

Substituting equation (38) into equation (36) gives,

$$z_k\, y_k = L_k \quad , \tag{40}$$

where,

$$z_k = (I - A\ \ r_{k,k} - r_{k+1,\ k}\ A\, E_k\, r_{k,\ k+1}\, A) \ , \tag{41}$$

$$L_k = r_{k+1,k} A\, E_k\left(m_{k+1} + A\left(\sum_{j=1}^{k-1} r_{j,k+1} y_j\right)\right) + m_k + A\left(\sum_{j=1}^{k-1} r_{j,k} y_j\right) . \tag{42}$$

Equation (40) can be solved using Gaussian elimination with partial pivoting.

At this point, the recursive algorithm for evaluating X can be summarized as follows:

Step 1 : Compute an orthogonal matrix V such that VT W V is a quasi-upper triangular by using the real Schur from decomposition.

Step 2 : If $r_{k+1,\,k} = 0$, hence compute y_k for k = 1, 2, ... , m from equation (35), else compute y_k from equation (40).

Step 3 : Obtain the desired solution matrix X from the relation $X = YV^T$.

Example 1

Considered the linear time-delay system described by the following state equation[13].

$$x(t) = 4x(t - 0.25) ,$$

with x(0) = 1.0 and x(t) = 0 for $-0.25 \le t < 0$. For q = 4, the PLPF of x(t) is solved for $0 \le t \le 1$. The computational results are given in Table 1. A comparison of the results by using the block-pulse functions, Taylor series approximation, and the exact solution are also presented in Table 1. From the table, it can be seen that very satisfactory results are obtained and the proposed method is more accurate than that of the block-pulse functions and Taylor series approximation.

TABLE 1. Solution of example 1.

Time (t)	Block-pulse method	Taylor series method	Proposed method	Exact
0	1.000	1.000	1.000	1.000
0.25	1.500	1.000	1.000	1.000
0.5	2.750	1.757	2.000	2.000
0.75	4.875	3.000	3.500	3.500
1.00	7.000	4.892	6.250	6.167

Parameter Identification

The identification problem is to estimate the unknown matrices A and B provide that x(0) ϕ_i, x_i and u_i are known. The system order n and the time delay α are assumed known a priori.

Now, by dropping P(t) from equation (25) gives:

$$X - K L = A \phi H P(\alpha) L + A X H[I - P(\alpha)L] + B U G \quad , \tag{43}$$

or

$$A N + B U G = X - K L \quad , \tag{44}$$

where,

$$N = \phi H P(\alpha) L + X H [I - P(\alpha)L] \quad , \tag{45}$$

By defining a parameter vector as,

$$q = [A \;\; B]T \quad , \tag{46}$$

we can rewrite equation (44) in compact form as ,

$$\Omega \theta = J1 \quad , \tag{47}$$

where,

$$\Omega = [N^T \quad (UG)^T] \ , \tag{48}$$

$$J1 = [X - KL]^T \ . \tag{49}$$

Since both W and J1 are known, the least-square estimate θ can be obtained as,

$$\theta = (\Omega^T \Omega)^{-1} \Omega^T J1 \ . \tag{50}$$

Provided that the matrix inversion exists and m ≥ (n + r). An illustrative example is given next which shows that the method gives accurate parameter estimates.

Example 2

Consider a system modelled by the following delay-differential equation

$$\dot{x}(t) = a\, x(t - 0.25) + b\, u(t)$$

with x(t) = 0 for t ≤ 0. The following numerical values are used.

$$u(t) = \text{unit impulse}$$

$$X \ [0 \quad 0.125 \quad 0.1875 \quad 0.3438 \quad 0.6094]$$

by using the proposed method, the estimated θ using q = 4 s

$$a = 4.0$$

$$b = 1.0$$

Notice that the data in this example are obtained with parameter values a = 4 and b = 1.

5. Conclusion

The PLPF has been successfully extended to solve the linear time-invariant delayed system. This approach is simple, straight forward and gives a piecewise point result. The recursive algorithm presented has two main advantages; first, the analytical requirements for both the input and output are not required. Second, tedious iterative algorithms and direct matrix inversion for large scale systems are avoided. These facts result in considerable saving of computing time. The results obtained are satisfactory.

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تحليـل وتعـريف نـظم خطيـة متأنيـة عن طريق دوال قطعيـة خطيـة متعـددة الحـــدود

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